3-D Parabolic PDEs – 2D Heat Equation

CPS 5310 Spring 2015

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Heat Conduction in a 2-D Domain

$$\frac{\partial u}{\partial t} = a^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \xi_x \frac{\partial u}{\partial x} + \xi_y \frac{\partial u}{\partial y} - \gamma u + f(x, y, t)$$

$$0 < x < l_x, \quad 0 < y < l_y, \quad t > 0$$

where $a^2 = k/c\rho$ is the thermal diffusivity; $\gamma = h/c\rho$ where h is the the heat exchange coefficient (for lateral heat exchange with an external medium; and $f(x,y,t) = Q(x,y,t)/c\rho$ where Q is the density of heat source or sink.

Initial condition: $u(x,y,0) = \phi(x,y)$

Boundary Conditions

$$\alpha_1 u_x + \beta_1 u \Big|_{x=0} = g_1(y,t)$$

$$\alpha_2 u_x + \beta_2 u \Big|_{x=l_x} = g_2(y,t)$$

$$\alpha_3 u_y + \beta_3 u \Big|_{y=0} = g_3(x,t)$$

$$\alpha_4 u_y + \beta_4 u \Big|_{y=l_y} = g_4(x,t)$$

Example 1

- The initial temperature distribution within a thin uniform rectangular plate with thermally insulated lateral faces is

$$u(x, y, 0) = Axy(l_x - x)(l_y - y)$$

where A is a constant.

- Find the distribution of temperature within the plate at any later time if its boundary is kept at a constant zero temperature. Generation or absorption of heat by external sources is absent.

Example 2

- Find the temperature $u(x,y,t)$ of a thin rectangular plate if its boundary is kept at constant zero temperature, the initial temperature distribution within the plate is zero, and one internal source of heat with value $Q(t) = A \sin \omega t$ acts at the point (x_0, y_0). Assume the plate is thermally insulated over its lateral surfaces.

Example 3

- A heat-conducting, thin, uniform rectangular plate is thermally insulated over its lateral faces. One side of the plate, at x = 0, is thermally insulated and the rest of the boundary is kept at constant zero temperature. The initial temperature distribution within the plate is zero. Let heat be generated throughout the plate with the intensity of internal sources (per unit mass of the membrane) given by

$$Q(x,y,t) = A(l_x - x)\sin\frac{\pi y}{l_y}\sin t$$

- Find the temperature distribution within the plate for t > 0.