Hyperbolic PDE – The Wave Equation

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1D Wave Equation

- Consider small transverse oscillations of a thin, stretched string.
- Let $u(x,t)$ give the string's amplitude at location x and time t.
- Assume $u_x(x,t)$ is small and $(u_x)^2 << 1$

Derivation of the Wave Equation

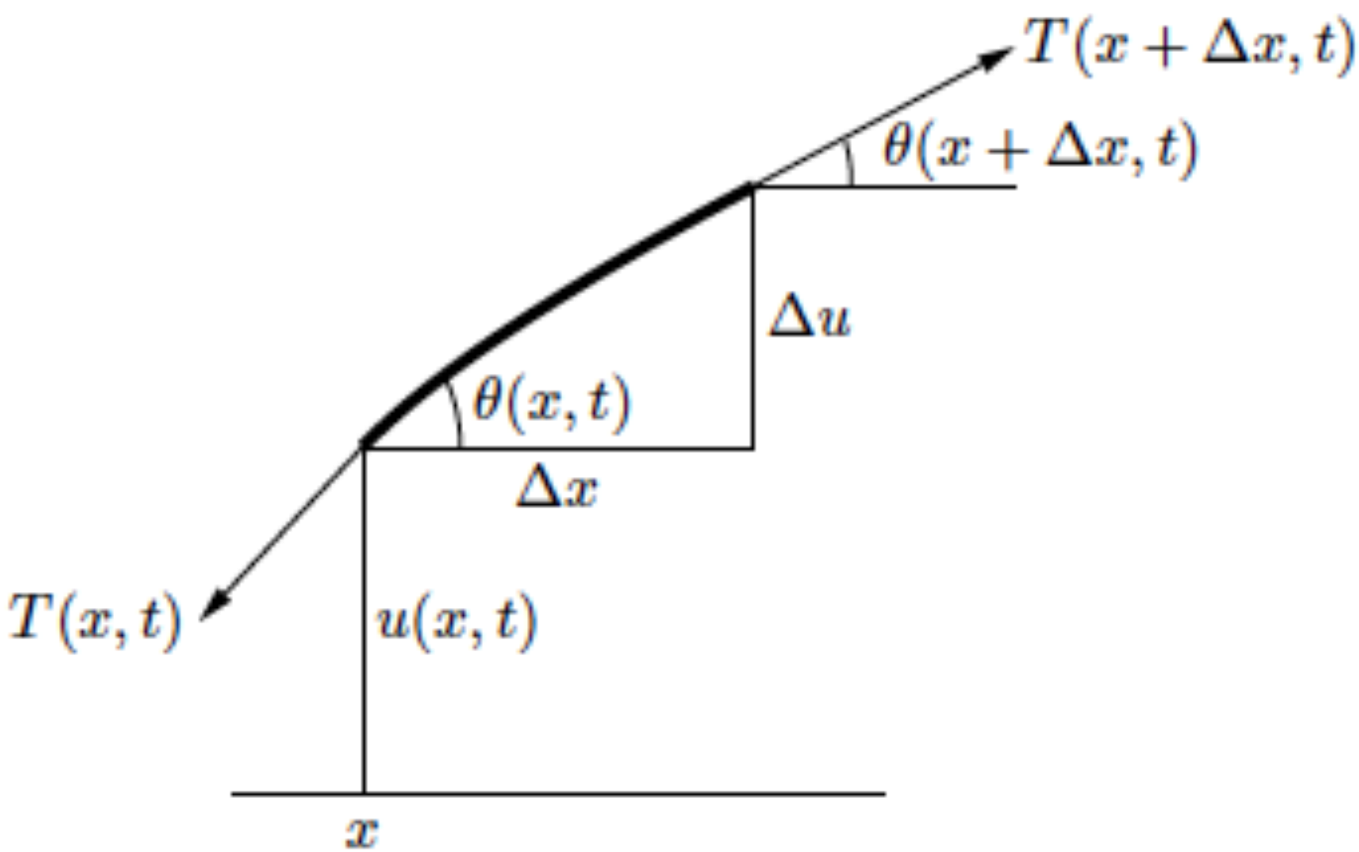

Image credit and derivation: http://www.math.ubc.ca/~feldman/apps/wave.pdf

Derivation (cont.)

- By considering the different forces on the string, we obtain

$$\frac{\partial^2 u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

- If the weight of the string can be neglected and there are no other external forces, we have the homogeneous wave equation for free oscillations of a string

$$\frac{\partial^2 u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Initial Conditions

- Since the equation is second order in time, we need two initial conditions.

$$u(x,0) = \varphi(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \psi(x)$$

Boundary Conditions

$$\alpha_1 u_x + \beta_1 u \Big|_{x=0} = g_1(t)$$

$$\alpha_2 u_x + \beta_2 u \Big|_{x=l} = g_2(t)$$

Can be Dirichlet, Neumann, or mixed

Example 1

- The ends of a uniform string of length l are fixed and all external forces including the gravitational force can be neglected. Displace the string from equilibrium by shifting the point $x = x_0$ by distance A at time $t = 0$ and then release it with zero initial speed. Find the displacements $u(x, t)$ of the string for times $t > 0$.

Example 2

- Consider a homogeneous string of mass density ρ with rigidly fixed ends. Starting at time $t = 0$ a uniformly distributed harmonic force with linear density $F(x,t) = F_0 \sin \omega t$ acts on the string. The initial deflection and speed are zero. Neglecting friction, find the resulting oscillations and investigate the resonance behavior.

Example 3

- The left end of a string is moving according to $g(t) = A \sin \omega t$ and the right end $x = l$ is secured. Initially the string is at rest. Describe oscillations when there are no external forces and the resistance of the surrounding medium is zero.