Numerical Solution of PDEs

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Symmetry and Dimensionality

- Numerical algorithms are extremely expensive in terms of computation time and memory requirements.
- So, simplify the problem first if possible.
- Main idea: Use symmetry to reduce the problem to one with lower dimensionality.
- Computational effort depends dramatically on the number of dimensions.

1D Example

- Consider a unit cube with
  - a constant temperature T_t at the top surface ($z = 1$)
  - a constant temperature T_b at the bottom surface ($z = 0$)
  - a perfect thermal insulation of all other surfaces of the cube
- What is the stationary temperature distribution $T(x, y, z)$ within the cube.
- Formulate the problem with as low a dimensionality as possible.

2D Example

- Consider a unit cube with
  - a constant temperature T_b at the back surface of the cube ($y = 1$)
  - a constant temperature T_f at the strip

    $y = 0$, $0.4 \leq x \leq 0.6$ at the front surface of the cube
  - a perfect insulation of all other surfaces of the cube
- What is the stationary temperature distribution within the cube?
- Formulate the problem with a low dimensionality as possible.

Rotational Symmetry

- Consider a cylinder with
  - a constant temperature T_t at the top surface of the cylinder
  - a constant temperature T_s at a strip around the cylinder
  - perfect thermal insulation of all other surfaces of the cylinder
- What is the stationary temperature distribution $T(x, y, z)$ within the cylinder?

Numerical Methods for Solving PDEs

- Finite difference methods
- Finite element methods
- Method of lines
- Spectral methods
- Finite volume methods

We will study the first two.

Finite Difference Methods

- Main idea: replace derivatives with finite differences
- 1D heat equation

$$\frac{\partial T}{\partial t} = \frac{K}{C\rho}\frac{\partial^2 T}{\partial x^2}$$

- Fourth-order central-difference approximation

$$\frac{T(x,t+\Delta t) - T(x,t)}{\Delta t} = \frac{K}{C\rho}\frac{T(x+\Delta x,t) + T(x-\Delta x,t) - 2T(x,t)}{(\Delta x)^2}$$

- Implement as

$$T_{i,j+1} = T_{i,j} + \eta\,(T_{i+1,j} + T_{i-1,j} - 2T_{i,j})$$

R Implementation

- Book software HeatClos.r
- Saves memory by storing temperatures at only two times
- Compares numerical and analytical solutions

Stability

- Try increasing Δt in HeatClos.r
- Beyond a certain limit, the solution becomes unstable.
- How can use choose Δt and Δx so that the solution is stable?
- HeatClos.r implements an explicit method
- Implicit methods (also called backwards methods) are more stable but require solving a system of (sparse) linear equations.