Elliptic PDEs

CPS 5310 Spring 2014

Shirley Moore, Instructor

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Laplace's Equation

- Many stationary (time-independent) processes are described by Laplace's equation:

$$\nabla^2 u = 0$$

- A function is said to be harmonic on a region if it satisfies Laplace's equation on the region and its first and second derivatives are continuous in that region. Examples: $2xy$, $x^2 - y^2$, $e^{-x} \cos y$
- Important in many fields of science, notably electromagnetism, astronomy, and fluid dynamics, because it can be used to accurately describe the behavior of electric, gravitational, and fluid potentials.
- In the study of heat conduction, the Laplace equation is the steady-state heat equation.

Poisson's Equation

- The nonhomogeneous equation

$$\nabla^2 u = -f$$

- Broad utility in electrostatics, mechanical engineering and theoretical physics
- Commonly used to model diffusion

Laplace Operator in Other Coordinates

- Polar coordinates:

$$x = r\cos\varphi, \quad y = r\sin\varphi$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

- Cylindrical coordinates:

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

Other Coordinates (cont.)

- Spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\theta^2}$$

Boundary Value Problems

- No initial conditions since problems are steady-state
- Can have Dirichlet, Neumann, and/or mixed boundary conditions that can be homogenous or non-homogeneous.