Parabolic PDEs: Heat Conduction and Diffusion

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Heat Conduction

- Flow of heat through a body due to a difference in temperature
- Kinetic process at the molecular level that involves energy transfer due to molecular collisions
- The heat equation models heat transfer in solids where there is no macroscopic mass transfer.

Heat (Conduction) Equation

$$\frac{\partial T}{\partial t} = \chi \, \Delta T$$ where $$\chi = \frac{K}{C\rho}$$ is the thermal diffusivity.

In a steady state, the temperature in the solid becomes time-independent and the heat equation reduces to Laplace's equation:

$$\Delta T = 0$$

If Q is the rate at which heat is added (or removed) per unit time and unit volume, the heat equations becomes

$$\frac{\partial T}{\partial t} = \chi \, \Delta T + \frac{Q}{C\rho}$$

Diffusion

- Diffusion is a mixing process that occurs when one substance is introduced into a second substance.
- The introduced substance spreads from where its concentration is higher to where its concentration is lower.
- Given sufficient time, diffusion will lead to equalizing of the concentration
  - unless the second substance is continually added at a specific location

Fick’s Law

- For low concentration gradients, diffusion obeys Fick’s law which states that the diffusion flux of a substance (amount of the substance that will flow through a small area during a small time interval) is inversely proportional to the concentration gradient:

$$I = D \cdot \nabla\Phi$$

where *D* is the coefficient of diffusion.

Diffusion Equation

- Combining Fick's Law with conservation of mass yields the diffusion equation:

$$\frac{\partial \Phi}{\partial t} = D \, \Delta \Phi$$

- The steady state equation is $\Delta \Phi = 0$
- If the introduced substance is being created or destroyed as described by function f:

$$\frac{\partial \Phi}{\partial t} = D \, \Delta \Phi + f$$

Diffusion with Disintegration or Multiplication

- Particles of the diffusing substance may be unstable in that they may disappear (e.g., as in an unstable gas or a gas being absorbed) or multiply (as with neutron diffusion).
- If the rates of these processes are proportional to the concentration, the process is described by $\frac{\partial \Phi}{\partial t} = D\Delta\Phi + \beta\Phi$ where β is the coefficient of disintegration or multiplication.

General Form of the 1D Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{dx^2} + \xi \frac{\partial u}{\partial x} - \gamma u + f(x,t)$$

with initial condition

$$u(x,0) = \varphi(x), \quad 0 < x < L$$

Boundary Conditions

- The general form of the boundary conditions is

$$P_1[u] \equiv \alpha_1 \frac{\partial u}{\partial x} + \beta_1 u \Big|_{x=0} = g_1(t)$$

$$P_2[u] \equiv \alpha_2 \frac{\partial u}{\partial x} + \beta_2 u \Big|_{x=L} = g_2(t)$$

- If α=0, it is a Dirichlet condition.
- If β=0, it is a Neumann condition.
- If both are nonzero, it is a mixed condition.
- If g(t)=0, it is a homogenous condition.