Partial Differential Equations (PDEs)

CPS 5310 Spring 2014

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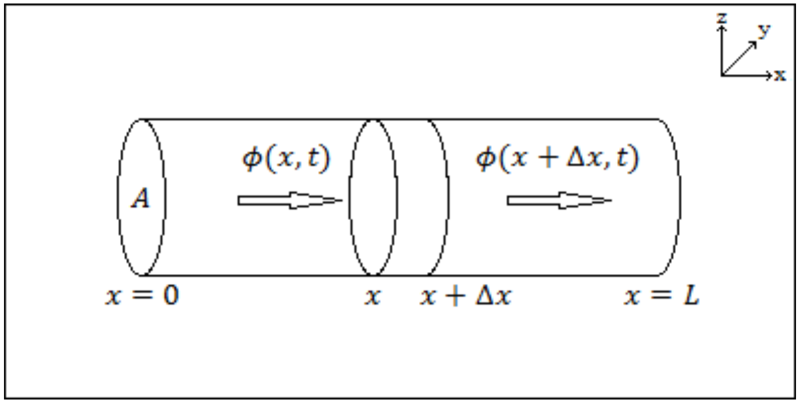
Overview

- ODEs model have limitations: Deviations between an ODE model and data may indicate dependence on more than one variable (e.g., time and space variables).
- PDE models involve derivatives with respect to at least two independent variables.
- There are many different types of PDEs, with each type requiring specific numerical techniques for solution.

Motivating Example: The Heat Equation

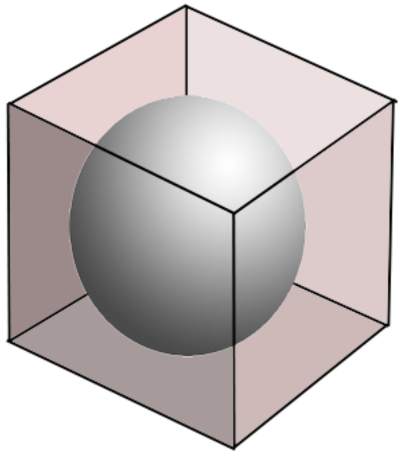
- Provides a way to compute temperature distributions
- Describes dynamics of temperature as a function of space and time
- Relevant to science and engineering since many processes are affected by temperature

Problem 1



- Assume a metallic cylinder with
  - perfect insulation of the cylinder surface
  - constant temperature in the y and z directions
  - a known initial temperature distribution $T_i(x)$ at time t=0
  - constant temperatures T_0 and T_L at the left and right ends of the cylinder for all times t
- What is the temperature $T(x,t)$ for $0 \le x \le L$ and $0 \le t \le t_{max}$?

Problem 2



- Assume a unit cube containing a sphere with
  - a constant temperature T_c at the top surface of the cube
  - a constant temperature T_s at the sphere surface
  - a perfect insulation of all other surfaces of the cube
- What is the stationary temperature T(x,y,z) within the cube (i.e., in the domain $[0,1]^3 \backslash S$)

Form of the Heat Equation

$$\frac{\partial T}{\partial t} = \frac{K}{C\rho}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

where *K* (*W* $K^{-1}m^{-1}$) is the thermal conductivity,
C (*J* kg^{-1} K^{-1}) is the specific heat capacity, and
σ (kg m^{-3}) is the density, or

$$\frac{\partial T}{\partial t} = \frac{K}{C\rho}\Delta T$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator.

Derivation of the Heat Equation: Fourier's Law

$$\begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = -K \cdot \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix}$$

or

$$q = -K \cdot \nabla T$$

where ***q*** is the heat flow rate in W m^{-2}

The heat flow is negatively proportional to the temperature gradient.

- The specific heat capacity C is a measure of the amount of heat energy required to increase the temperature of 1 kg of the material by 1 °C.

Derivation of the Heat Equation: Conservation of Energy

- Consider a small interval $[x, x + \Delta x]$ corresponding to a small part of the cylinder from Problem 1.
- The principle of conservation of energy states that energy cannot be created nor destroyed – i.e., the total amount of energy in any closed system remains constant.
- Any change in the energy content of $[x, x + \Delta x]$ must equal the amount that flows through its ends at x and $x + \Delta x$.

Heat Equation = Fourier's Law + Conservation of Energy

Anisotropic Case

- When thermal conductivity is direction-dependent
- K is a matrix in this case

Types of PDEs

- The general form of a linear second-order PDE is two dimensions is

  $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_{x} + Eu_{y} + F = 0$
- The discriminant $d = AC – B^2$
- The PDE is
  - elliptic if $d > 0$
  - parabolic if $d = 0$
  - hyperbolic if $d < 0$